

# Mappings from $\mathbb{R}^3$ to $\mathbb{R}^3$ and signs of swallowtails

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## 1 Introduction

Let  $M$  be an oriented 3-manifold. According to [2], for a residual set of mappings  $f \in C^\infty(M, \mathbb{R}^3)$ , the 3-jet  $j^3 f$  intersects transversely the Boardman strata  $S_1$ ,  $S_{1,1}$ ,  $S_{1,1,1}$  in  $J^3(M, \mathbb{R}^3)$ . If that is the case then  $S_{1,1,1}(f)$  is a discrete subset of  $M$ .

According to Morin [10], if  $j^3 f$  is also transversal to  $S_{1_3}$  then  $S_{1,1,1}(f) = S_{1_3}(f)$  and any  $p \in S_{1,1,1}(f)$  is a swallowtail point, so that there exists a well-oriented coordinate system  $x, y, z$  centered at  $p$ , and a coordinate system centered at  $f(p)$  such that  $f$  has the form  $f_\pm(x, y, z) = (\pm xy + x^2 z + x^4, y, z)$ . Hence one may associate a sign with  $p \in S_{1,1,1}(f)$ . A geometric definition of the sign associated with a swallowtail was originally introduced by Goryunov in [5].

In this paper we give a definition of a simple swallowtail point  $p \in S_{1,1,1}(f)$ , and define its sign  $I(f, p)$ . We shall show (see Theorem 7.1) that

- if  $p$  is a simple swallowtail then  $j^3 f$  intersects transversely  $S_{1_3}$  at  $p$ ,
- $I(f, p) = +1$  (resp.  $-1$ ) if and only if  $f$  has the form  $f_+$  (resp.  $f_-$ ).

If every  $p \in S_{1,1,1}(f)$  is a simple swallowtail and  $S_{1,1,1}(f)$  is finite, then numbers  $\#S_{1,1,1}(f)$ ,  $\#\{p \in S_{1,1,1}(f) \mid I(f, p) = \pm 1\}$  are important invariants associated with  $f$ . (We refer the reader to [5, 12], where the first-order invariants of stable mappings in  $C^\infty(M, \mathbb{R}^3)$  are classified. Articles [9, 11] present classification of some families of germs from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ .) We shall show how to compute these numbers in the case where  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is a polynomial mapping, in terms of signatures of quadratic forms.

It is proper to add that in [6] there is presented a construction of quadratic forms whose signatures determine the number of positive and negative cusps of a generic polynomial mapping from the plane to the plane.

## 2 Preliminaries

In this section we have compiled some basic facts on singularities of mappings between manifolds. The best reference here is [4].

Let  $X, Y$  be finite-dimensional smooth manifolds. If  $p$  is a point in  $X$  then  $E(p)$  denotes the local ring of germs of smooth functions at  $p$ . Its maximal ideal  $m(p)$  consists of germs of functions vanishing at  $p$ . A map germ  $f : (X, p) \rightarrow (Y, q)$  induces a homomorphism of local rings  $f^* : E(q) \rightarrow E(p)$ . The local ring of  $f$  at  $p$  is the quotient ring  $\mathcal{R}_f(p) = E(p)/E(p)f^*(m(q))$ .

We say that  $f$  takes on a Morin singularity of type  $k$  at  $p$  if  $\mathcal{R}_f(p) \cong \mathbb{R}[t]/(t^{k+1})$  for some  $k$ .

Let  $J^k(X, Y)$  be the  $k$ -jet bundle over  $X \times Y$ . According to [10] and [7, 8], if  $\dim X = \dim Y$  then there exists a submanifold  $S_{1_k}$  of  $J^k(X, Y)$  of codimension  $k$  such that for a smooth  $f : X \rightarrow Y$  the set  $S_{1_k}(f) = (j^k f)^{-1}(S_{1_k})$  consists of points where  $f$  takes on a Morin singularity of type  $k$ . For a residual set of mappings  $f \in C^\infty(X, Y)$ ,  $S_{1_k}(f)$  is a submanifold of  $X$  of codimension  $k$ .

For a mapping  $f : X \rightarrow Y$ , a point  $p$  is in  $S_1(f)$  if  $f$  has corank 1 at  $p$ . A point  $p \in S_1(f)$  is in  $S_{1,1}(f)$  if  $f|_{S_1(f)}$  has corank 1, assuming that  $S_1(f)$  is a submanifold. A point  $p \in S_{1,1}(f)$  is in  $S_{1,1,1}(f)$  if  $f|_{S_{1,1}(f)}$  has corank 1, assuming that  $S_{1,1}(f)$  is a submanifold.

According to Boardman [2], for a residual set of mappings  $f \in C^\infty(X, Y)$ , sets  $S_1(f)$ ,  $S_{1,1}(f)$ , and  $S_{1,1,1}(f)$  are submanifolds.

## 3 Sign of a swallowtail

Let  $f_\pm = (\pm xy + x^2z + x^4, y, z) : (\mathbb{R}^3, \mathbf{0}) \rightarrow (\mathbb{R}^3, \mathbf{0})$ .

**Example 3.1.** There exists an orientation reversing diffeomorphism  $\phi_- = (-x, y, z) : (\mathbb{R}^3, \mathbf{0}) \rightarrow (\mathbb{R}^3, \mathbf{0})$  such that  $f_- = f_+ \circ \phi_-$ .

**Example 3.2.** There exist an orientation preserving diffeomorphism  $\phi_+ = (-x, -y, z) : (\mathbb{R}^3, \mathbf{0}) \rightarrow (\mathbb{R}^3, \mathbf{0})$  and an orientation reversing diffeomorphism  $\psi_- = (x, -y, z) : (\mathbb{R}^3, \mathbf{0}) \rightarrow (\mathbb{R}^3, \mathbf{0})$  such that  $f_+ = \psi_- \circ f_+ \circ \phi_+$ .

Let  $f : (\mathbb{R}^3, p_0) \rightarrow (\mathbb{R}^3, f(p_0))$  be a smooth mapping defined in a neighbourhood of  $p_0$ , and let  $J = \det[Df]$ . Denote  $S_1(f) = \{p \in \mathbb{R}^3 \mid \text{rank } Df(p) = 2\}$ .

**Remark 3.3.** Assume that  $\text{rank}[Df(p_0)] = 2$  and the gradient  $\nabla J(p_0) \neq \mathbf{0}$ . Then  $J(p_0) = 0$  and the set  $J^{-1}(0)$  of critical points of  $f$  is locally a smooth surface. Moreover  $J^{-1}(0) = S_1(f)$  near  $p_0$ . Hence  $\dim \text{Ker } Df(p) = 1$  along  $S_1(f)$ .

From now on we shall suppose that assumptions of the above remark hold. Let  $K : (\mathbb{R}^3, p_0) \rightarrow \mathbb{R}^3$  be a vector field such that  $K(p_0) \neq \mathbf{0}$  and  $Df(p) K(p) \equiv \mathbf{0}$  for  $p \in S_1(f)$ , so that  $K$  is in the kernel of  $Df$  along  $S_1(f)$ .

**Lemma 3.4.** If  $f = (f_1, f_2, f_3)$  then at least one vector product  $\nabla f_i \times \nabla f_j$  satisfies above conditions.

*Proof.* Since  $\text{rank}[Df(p_0)] = 2$ , two rows of the derivative matrix are linearly independent. Hence at least one  $2 \times 2$ -minor is non zero, and then there exist  $1 \leq i < j \leq 3$  with  $(\nabla f_i \times \nabla f_j)(p_0) \neq \mathbf{0}$ .

Let  $p \in S_1(f) = J^{-1}(0)$ . Notice that one of the coordinates of the composition  $Df(p) (\nabla f_i \times \nabla f_j)(p)$  equals  $\pm J(p) = 0$ . The other two coordinates are equal to the Laplace expansion of matrices having the same two rows, so they vanish too. So  $Df(p) (\nabla f_i \times \nabla f_j)(p) \equiv \mathbf{0}$ .  $\square$

Put  $X = \langle \nabla J, K \rangle$ , where  $\langle \cdot, \cdot \rangle$  denotes the standard scalar product. Denote  $S_{1,1}(f) = \{p \in S_1(f) \mid \text{rank } D(f|_{S_1(f)})(p) = 1\}$ .

**Remark 3.5.** As  $\dim \text{Ker } Df = 1$  along  $S_1(f)$ , then the set of critical points of  $f|_{S_1(f)}$  equals  $S_{1,1}(f)$ . Suppose that  $X(p_0) = 0$  and vectors  $\nabla J(p_0), \nabla X(p_0)$  are linearly independent. Then locally

$$\begin{aligned} S_{1,1}(f) &= \{p \in S_1(f) \mid K(p) \in T_p S_1(f)\} = \{p \in J^{-1}(0) \mid K(p) \in T_p J^{-1}(0)\} \\ &= \{p \in J^{-1}(0) \mid K(p) \perp \nabla J(p)\} = J^{-1}(0) \cap X^{-1}(0). \end{aligned}$$

Moreover  $S_{1,1}(f)$  is locally a smooth curve near  $p_0$ .

From now on we shall suppose that assumptions of the above remark hold. Put  $Y = \langle \nabla X, K \rangle$ . Denote  $S_{1,1,1}(f) = \{p \in S_{1,1}(f) \mid \text{rank } D(f|_{S_{1,1}(f)})(p) = 0\}$ .

**Remark 3.6.** As  $S_{1,1}(f)$  is one-dimensional, then the set of critical points of  $f|_{S_{1,1}(f)}$  equals  $S_{1,1,1}(f)$ . Suppose that  $Y(p_0) = 0$ . Then locally

$$S_{1,1,1}(f) = \{p \in S_{1,1}(f) \mid K(p) \in T_p(S_{1,1}(f))\}$$

$$\begin{aligned}
&= \{p \in J^{-1}(0) \cap X^{-1}(0) \mid K(p) \in T_p(J^{-1}(0)) \cap T_p(X^{-1}(0))\} \\
&= \{p \in J^{-1}(0) \cap X^{-1}(0) \mid K(p) \perp \nabla J(p), K(p) \perp \nabla X(p)\} \\
&= J^{-1}(0) \cap X^{-1}(0) \cap Y^{-1}(0).
\end{aligned}$$

**Definition 3.7.** We shall say that  $p_0$  is a simple swallowtail point if  $\text{rank}[Df(p_0)] = 2$ ,  $J(p_0) = X(p_0) = Y(p_0) = 0$ , and vectors  $\nabla J(p_0), \nabla X(p_0), \nabla Y(p_0)$  are linearly independent. In that case we say that the germ  $f$  is a simple swallowtail

If that is the case then the vectors  $\nabla J(p_0), \nabla X(p_0), \nabla Y(p_0)$  are linearly independent, so  $\{p_0\}$  is an isolated point in  $S_{1,1,1}(f)$ . Put  $Z = \langle \nabla Y, K \rangle$ . Since  $X(p_0) = Y(p_0) = 0$ , the vector  $K(p_0)$  is perpendicular to both  $\nabla J(p_0)$  and  $\nabla X(p_0)$ . Hence  $Z(p_0) = \langle \nabla Y(p_0), K(p_0) \rangle \neq 0$ .

**Corollary 3.8.** Let  $H = (J, X, Y)$ . The point  $p_0$  is a simple swallowtail if and only if  $\text{rank}[Df(p_0)] = 2$  and  $H$  has a simple zero at  $p_0$ . If that is the case then  $Z(p_0) \neq 0$ .  $\square$

**Definition 3.9.** If the germ  $f : (\mathbb{R}^3, p_0) \rightarrow (\mathbb{R}^3, f(p_0))$  is a simple swallowtail then its sign, denoted by  $I(f, p_0)$ , is defined to be  $I(f, p_0) = \text{sgn}(Z(p_0)) \cdot \det[DH(p_0)]$ , i.e.

$$I(f, p_0) = \text{sgn}(Z(p_0)) \cdot \text{sgn}(\det[\nabla J(p_0), \nabla X(p_0), \nabla Y(p_0)]).$$

**Example 3.10.** Let  $f = f_{\pm} = (\pm xy + x^2z + x^4, y, z)$ ,  $p_0 = \mathbf{0}$ . Then

$$Df = \begin{bmatrix} \pm y + 2xz + 4x^3 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$J = \pm y + 2xz + 4x^3, \quad K = (1, 0, 0),$$

$$X = 2z + 12x^2, \quad Y = 24x, \quad Z = 24.$$

Because

$$\det[DH(\mathbf{0})] = \det \begin{bmatrix} 0 & \pm 1 & 0 \\ 0 & 0 & 2 \\ 24 & 0 & 0 \end{bmatrix} = \pm 48,$$

then both  $f_{\pm}$  are simple swallowtails and  $I(f_{\pm}, \mathbf{0}) = \pm 1$ .

In the next sections we shall show that

- (1)  $I(f, p_0)$  does not depend on the choice of the field  $K$ ,
- (2) if  $\phi, \psi$  are germs of diffeomorphisms and  $\phi$  preserves the orientation then the sign of  $f$  equals the sign of  $\psi \circ f \circ \phi$ ,
- (3) there exist germs of diffeomorphisms  $\phi, \psi$  such that  $\phi$  preserves the orientation and  $\psi \circ f \circ \phi = f_+$  (resp.  $= f_-$ ) if  $I(f, p_0) = +1$  (resp.  $-1$ ).

## 4 On the choice of the field $K$

In this section we shall show that the sign of a swallowtail does not depend on the choice of the field  $K$ .

Suppose that assumptions of Remark 3.3 hold and  $K : (\mathbb{R}^3, p_0) \rightarrow \mathbb{R}^3$  is the vector field introduced in the previous section.

Let  $K_1 : (\mathbb{R}^3, p_0) \rightarrow \mathbb{R}^3$  be a vector field such and  $Df(p) K_1(p) \equiv \mathbf{0}$  for  $p \in S_1(f)$ , so that  $K_1$  is in the kernel of  $Df$  along  $S_1(f)$ . (We do not exclude the case where  $K_1(p_0) = \mathbf{0}$ .) The kernel of  $Df$  along  $S_1(f)$  is one-dimensional, so fields  $K$  and  $K_1$  are collinear along  $S_1(f)$ , and there exists a smooth  $\xi : (\mathbb{R}^3, p_0) \rightarrow \mathbb{R}$  such that  $K_1(p) = \xi(p) \cdot K(p)$  for  $p \in S_1(f)$ . Since  $S_1(f) = J^{-1}(0)$  is locally a complete intersection, there is  $L : (\mathbb{R}^3, p_0) \rightarrow \mathbb{R}^3$  with  $K_1 = \xi \cdot K + J \cdot L$ .

Take any smooth  $\alpha : (\mathbb{R}^3, p_0) \rightarrow \mathbb{R}$ , and set  $J_1 = \alpha \cdot J$ . Of course,  $J_1(p_0) = 0$ . Put  $X_1 = \langle \nabla J_1, K_1 \rangle$ ,  $Y_1 = \langle \nabla X_1, K_1 \rangle$ . We have

$$X_1 = \langle \nabla(\alpha \cdot J), K_1 \rangle = \langle \alpha \cdot \nabla J + J \cdot \nabla \alpha, \xi \cdot K + J \cdot L \rangle = (\alpha \cdot \xi) \cdot X + J \cdot R_1,$$

$$\begin{aligned} Y_1 &= \langle \nabla((\alpha \cdot \xi) \cdot X) + \nabla(J \cdot R_1), K_1 \rangle \\ &= \langle (\alpha \cdot \xi) \nabla X + X \cdot \nabla(\alpha \cdot \xi) + J \cdot \nabla(R_1) + R_1 \cdot \nabla J, \xi \cdot K + J \cdot L \rangle \\ &= (\alpha \cdot \xi^2) Y + R_2 \cdot J + (\langle \nabla(\alpha \cdot \xi), K_1 \rangle + R_1 \cdot \xi) \cdot X \\ &= (\alpha \cdot \xi^2) Y + R_2 \cdot J + R_3 \cdot X, \end{aligned}$$

where  $R_1, R_2, R_3 : (\mathbb{R}^3, p_0) \rightarrow \mathbb{R}$ . Hence  $X_1(p_0) = 0$ ,  $Y_1(p_0) = 0$ , and

$$\begin{bmatrix} J_1 \\ X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} \alpha & 0 & 0 \\ R_1 & \alpha \cdot \xi & 0 \\ R_2 & R_3 & \alpha \cdot \xi^2 \end{bmatrix} \begin{bmatrix} J \\ X \\ Y \end{bmatrix}.$$

**Corollary 4.1.** *One has  $J^{-1}(0) \subset J_1^{-1}(0)$ ,  $J^{-1}(0) \cap X^{-1}(0) \subset J_1^{-1}(0) \cap X_1^{-1}(0)$ , and  $J^{-1}(0) \cap X^{-1}(0) \cap Y^{-1}(0) \subset J_1^{-1}(0) \cap X_1^{-1}(0) \cap Y_1^{-1}(0)$ .  $\square$*

Put  $H_1 = (J_1, X_1, Y_1) : (\mathbb{R}^3, p_0) \rightarrow (\mathbb{R}^3, \mathbf{0})$ .

**Lemma 4.2.** *We have  $H^{-1}(\mathbf{0}) \subset H_1^{-1}(\mathbf{0})$  near  $p_0$ . If  $\alpha(p_0)\xi(p_0) \neq 0$  then locally  $H^{-1}(\mathbf{0}) = H_1^{-1}(\mathbf{0})$ .  $\square$*

Let  $Z_1 = \langle \nabla Y_1, K_1 \rangle$ .

**Lemma 4.3.** *We have  $Z_1(p_0) = \alpha(p_0)\xi^3(p_0)Z(p_0)$ .*

*Proof.* Since  $J(p_0) = X(p_0) = Y(p_0) = 0$ , one has

$$\nabla Y_1(p_0) = \alpha(p_0)\xi^2(p_0)\nabla Y(p_0) + R_2(p_0)\nabla J(p_0) + R_3(p_0)\nabla X(p_0).$$

Then  $Z_1(p_0) = \langle \nabla Y_1(p_0), \xi(p_0)K(p_0) \rangle = \alpha(p_0)\xi^3(p_0)Z(p_0) + R_2(p_0)\xi(p_0)X(p_0) + R_3(p_0)\xi(p_0)Y(p_0) = \alpha(p_0)\xi^3(p_0)Z(p_0)$ .  $\square$

**Proposition 4.4.** *We have*

- (i)  $\nabla J_1(p_0)$  is a linear combination of  $\nabla J(p_0)$ ,  $\nabla X_1(p_0)$  is a linear combination of  $\nabla J(p_0), \nabla X(p_0)$ , and  $\nabla Y_1(p_0)$  is a linear combination of  $\nabla J(p_0), \nabla X(p_0), \nabla Y(p_0)$ ,
- (ii) if  $\alpha(p_0)\xi(p_0) \neq 0$  then  $p_0$  is a simple zero of  $H$  if and only if  $p_0$  is a simple zero of  $H_1$ . If that is the case then

$$\text{sgn}(Z_1(p_0) \cdot \det[DH_1(p_0)]) = \text{sgn}(Z(p_0) \cdot \det[DH(p_0)]) = I(f, p_0).$$

*In particular, the definition of  $I(f, p_0)$  does not depend on the choice of the vector field  $K$  with  $K(p_0) \neq \mathbf{0}$ . (In this case one may take  $\alpha \equiv 1$ .)*

- (iii) if  $\alpha(p_0)\xi(p_0) = 0$  then  $\text{sgn}(Z_1(p_0) \cdot \det[DH_1(p_0)]) = 0$ .

*Proof.* We have

$$\nabla J_1(p_0) = \alpha(p_0)\nabla J(p_0),$$

$$\nabla X_1(p_0) = R_1(p_0)\nabla J(p_0) + \alpha(p_0)\xi(p_0)\nabla X(p_0),$$

$$\nabla Y_1(p_0) = R_2(p_0)\nabla J(p_0) + R_3(p_0)\nabla X(p_0) + \alpha(p_0)\xi^2(p_0)\nabla Y(p_0).$$

$$\text{Then } \det[DH_1(p_0)] = \det[\nabla J_1(p_0), \nabla X_1(p_0), \nabla Y_1(p_0)]$$

$$= \alpha^3(p_0)\xi^3(p_0) \det[\nabla J(p_0), \nabla X(p_0), \nabla Y(p_0)] = \alpha^3(p_0)\xi^3(p_0) \det[DH(p_0)].$$

By Lemma 4.3, if  $\alpha(p_0)\xi(p_0) \neq 0$  then

$$\begin{aligned} \operatorname{sgn}(Z_1(p_0) \cdot \det[DH_1(p_0)]) &= \operatorname{sgn}(\alpha(p_0)\xi^3(p_0)Z(p_0) \cdot \alpha^3(p_0)\xi^3(p_0) \det[DH(p_0)]) \\ &= \operatorname{sgn}(Z(p_0) \cdot \det[DH(p_0)]). \end{aligned}$$

If  $\alpha(p_0)\xi(p_0) = 0$  then  $Z_1(p_0) \cdot \det[DH_1(p_0)] = 0$ .  $\square$

## 5 Coordinates in the domain

In this section we shall show that an orientation preserving change of coordinates in the domain does not change the sign of a swallowtail.

Suppose that the germ  $f : (\mathbb{R}^3, p_0) \rightarrow (\mathbb{R}^3, f(p_0))$  is a simple swallowtail. Let  $\phi : (\mathbb{R}^3, q_0) \rightarrow (\mathbb{R}^3, p_0)$  be a germ of a diffeomorphism. Put  $\tilde{f} = f \circ \phi$ . Then the derivative matrix  $[D\tilde{f}] = [Df(\phi)] \cdot [D\phi]$ , and then  $\tilde{J} = \det[D\tilde{f}] = J(\phi) \cdot \det[D\phi]$ . Hence  $J(\phi) = \tilde{\alpha} \cdot \tilde{J}$ , where  $\tilde{\alpha} = (\det[D\phi])^{-1}$ ,  $\tilde{\alpha}(q_0) \neq 0$ .

We have  $\nabla(J(\phi)) = [D\phi]^T \cdot \nabla J(\phi)$  and  $\nabla J(\phi)(q_0) = \nabla J(p_0) \neq \mathbf{0}$ . Therefore  $\nabla(J(\phi))(q_0) \neq \mathbf{0}$ . Since  $\tilde{J}(q_0) = 0$ , we have  $\nabla(J(\phi))(q_0) = \tilde{\alpha}(q_0) \cdot \nabla \tilde{J}(q_0)$ , and then  $\nabla \tilde{J}(q_0) \neq \mathbf{0}$ .

Denote  $S_1(\tilde{f}) = \{q \in \mathbb{R}^n \mid \operatorname{rank} D\tilde{f}(q) = 2\}$ . Since  $\operatorname{rank} D\tilde{f} = \operatorname{rank} Df(\phi)$ , the set  $\tilde{J}^{-1}(0)$  consisting of critical points of  $\tilde{f}$  equals  $\phi^{-1}(S_1(f))$ , and so is a smooth surface.

Put  $\tilde{K} = [D\phi]^{-1} \cdot K(\phi)$ . Then

$$D\tilde{f} \cdot \tilde{K} = Df(\phi) \cdot D\phi \cdot [D\phi]^{-1} K(\phi) = (Df \cdot K)(\phi) \equiv \mathbf{0}$$

along  $S_1(\tilde{f})$ . Moreover  $\tilde{K}(q_0) \neq \mathbf{0}$ .

Put  $\tilde{X} = \langle \nabla \tilde{J}, \tilde{K} \rangle$  and  $\tilde{X}_1 = \langle \nabla(J(\phi)), \tilde{K} \rangle$ . Then

$$\tilde{X}_1 = \langle [D\phi]^T \cdot \nabla J(\phi), [D\phi]^{-1} \cdot K(\phi) \rangle = \langle \nabla J(\phi), K(\phi) \rangle = X(\phi).$$

Put  $\tilde{Y} = \langle \nabla \tilde{X}, \tilde{K} \rangle$  and  $\tilde{Y}_1 = \langle \nabla \tilde{X}_1, \tilde{K} \rangle$ . Then

$$\tilde{Y}_1 = \langle \nabla(X(\phi)), \tilde{K} \rangle = \langle [D\phi]^T \cdot \nabla X(\phi), [D\phi]^{-1} \cdot K(\phi) \rangle = \langle \nabla X(\phi), K(\phi) \rangle = Y(\phi).$$

Take mappings  $\tilde{H} = (\tilde{J}, \tilde{X}, \tilde{Y})$  and  $\tilde{H}_1 = (J(\phi), X(\phi), Y(\phi)) = H(\phi)$ .

Put  $\tilde{Z} = \langle \nabla \tilde{Y}, \tilde{K} \rangle$  and  $\tilde{Z}_1 = \langle \nabla \tilde{Y}_1, \tilde{K} \rangle$ . Then  $\tilde{Z}_1 = \langle \nabla(Y(\phi)), \tilde{K} \rangle = \langle [D\phi]^T \cdot \nabla J(\phi), [D\phi]^{-1} \cdot K(\phi) \rangle = Z(\phi)$ , so that  $\tilde{Z}_1(q_0) = Z(p_0) \neq 0$ .

We have  $D\tilde{H}_1 = DH(\phi) \cdot D\phi$ , so that  $\tilde{H}_1$  has a simple zero at  $q_0$ . We may apply results of the previous section, where  $\xi \equiv 1$ , and we replace  $J$  by  $\tilde{J}$ ,  $\alpha$  by  $\tilde{\alpha}$ ,  $K$  by  $\tilde{K}$ ,  $X$  by  $\tilde{X}$ ,  $X_1$  by  $\tilde{X}_1$ , and so on. By Proposition 4.4 (ii) and Corollary 3.8,  $\tilde{H}$  has a simple zero at  $q_0$ ,  $\tilde{Z}(q_0) \neq 0$ , and

$$\begin{aligned} I(\tilde{f}, q_0) &= \text{sgn}(\tilde{Z}(q_0) \cdot \det[D\tilde{H}(q_0)]) = \text{sgn}(\tilde{Z}_1(q_0) \cdot \det[D\tilde{H}_1(q_0)]) \\ &= \text{sgn}(Z(p_0) \cdot \det[DH(p_0)] \cdot \det[D\phi(q_0)]) = I(f, p_0) \cdot \text{sgn} \det[D\phi(q_0)]. \end{aligned}$$

We have got

**Proposition 5.1.** *If  $\phi : (\mathbb{R}^3, q_0) \rightarrow (\mathbb{R}^3, p_0)$  is a germ of a diffeomorphism then  $f$  is a simple swallowtail if and only if  $f \circ \phi$  does. If that is the case then*

$$I(f \circ \phi, q_0) = I(f, p_0) \cdot \text{sgn} \det[D\phi(q_0)] . \quad \square$$

## 6 Coordinates in the target

In this section we shall show that a change of coordinates in the target does not change the sign of a swallowtail.

Let  $\psi : (\mathbb{R}^3, f(p_0)) \rightarrow (\mathbb{R}^3, s_0)$  be a germ of a diffeomorphism. Put  $\bar{f} = \psi \circ f$ . Then the derivative matrix  $[D\bar{f}] = [D\psi(f)] \cdot [Df]$ , and then  $\bar{J} = \det[D\bar{f}] = \det[D\psi(f)] \cdot J = \bar{\alpha} \cdot J$ , where  $\bar{\alpha} = \det[D\psi(f)]$ ,  $\bar{\alpha}(p_0) \neq 0$ . Since  $J(p_0) = 0$ , we have  $\bar{J}(p_0) = 0$  and  $\nabla \bar{J}(p_0) = \bar{\alpha}(p_0) \nabla J(p_0) \neq \mathbf{0}$ .

Denote  $S_1(\bar{f}) = \{p \in \mathbb{R}^n \mid \text{rank } D\bar{f}(p) = 2\}$ . Since  $\text{rank } D\bar{f} = \text{rank } Df$ , the set  $\bar{J}^{-1}(0)$  of critical points of  $\bar{f}$  equals  $S_1(\bar{f})$ , and so is a smooth surface. Of course  $D\bar{f} \cdot K = D\psi(f) \cdot (Df \cdot K) \equiv \mathbf{0}$  along  $S_1(\bar{f}) = S_1(f)$ .

Put  $\bar{X} = \langle \nabla \bar{J}, K \rangle$ . Hence  $\bar{X} = \langle \bar{\alpha} \cdot \nabla J + J \cdot \nabla \bar{\alpha}, K \rangle = \bar{\alpha} \cdot X + \bar{R}_1 \cdot J$ , where  $\bar{R}_1 = \langle \nabla \bar{\alpha}, K \rangle$ . In particular,  $\bar{X}(p_0) = 0$ . Moreover,  $\nabla \bar{X} = \bar{\alpha} \cdot \nabla X + X \cdot \nabla \bar{\alpha} + \bar{R}_1 \cdot \nabla J + J \cdot \nabla \bar{R}_1$ , so that  $\nabla \bar{X}(p_0) = \bar{\alpha}(p_0) \nabla X(p_0) + \bar{R}_1(p_0) \nabla J(p_0)$ .

$$\text{Put } \bar{Y} = \langle \nabla \bar{X}, K \rangle = \langle \bar{\alpha} \cdot \nabla X + X \cdot \nabla \bar{\alpha} + \bar{R}_1 \cdot \nabla J + J \cdot \nabla \bar{R}_1, K \rangle$$

$$= \langle \nabla \bar{R}_1, K \rangle J + (\langle \nabla \bar{\alpha}, K \rangle + \bar{R}_1) \cdot X + \bar{\alpha} \cdot Y = \bar{R}_2 \cdot J + \bar{R}_3 \cdot X + \bar{\alpha} \cdot Y .$$

In particular,  $\bar{Y}(p_0) = 0$ . Moreover

$$\nabla \bar{Y}(p_0) = \bar{R}_2(p_0) \nabla J(p_0) + \bar{R}_3(p_0) \cdot \nabla X(p_0) + \bar{\alpha}(p_0) \nabla Y(p_0) .$$

Since  $\bar{\alpha}(p_0) \neq 0$ , vectors  $\nabla \bar{J}(p_0)$ ,  $\nabla \bar{X}(p_0)$ ,  $\nabla \bar{Y}(p_0)$  are linearly independent.

Put  $\bar{H} = (\bar{J}, \bar{X}, \bar{Y})$  and  $\bar{Z} = \langle \nabla \bar{Y}, K \rangle$ . The mapping  $\bar{H}$  has a simple zero at  $p_0$  if and only if  $H$  does.



**Proposition 6.1.** *Let  $\psi : (\mathbb{R}^3, f(p_0)) \rightarrow (\mathbb{R}^3, s_0)$  be a germ of a diffeomorphism. Then  $f$  is a simple swallowtail if and only if  $\psi \circ f$  does. If that is the case then*

$$I(\psi \circ f, p_0) = I(f, p_0).$$

*Proof.* We have  $\bar{Z}(p_0) = \langle \nabla \bar{Y}(p_0), K(p_0) \rangle = \bar{R}_2(p_0) \cdot X(p_0) + \bar{R}_3(p_0) \cdot Y(p_0) + \bar{\alpha}(p_0) \cdot Z(p_0) = \bar{\alpha}(p_0) \cdot Z(p_0)$ . Using similar arguments as before one may observe that

$$\begin{aligned} I(\psi \circ f, p_0) &= \text{sgn}(\bar{Z}(p_0) \cdot \det[D\bar{H}(p_0)]) \\ &= \text{sgn}(\bar{\alpha}(p_0) \cdot Z(p_0) \cdot (\bar{\alpha}(p_0))^3 \cdot \det[DH(p_0)]) = I(f, p_0). \quad \square \end{aligned}$$

By Propositions 5.1 and 6.1 we get

**Theorem 6.2.** *Let  $\phi : (\mathbb{R}^3, q_0) \rightarrow (\mathbb{R}^3, p_0)$  and  $\psi : (\mathbb{R}^3, f(p_0)) \rightarrow (\mathbb{R}^3, s_0)$  be germs of diffeomorphisms. Then  $f$  is a simple swallowtail if and only if  $\psi \circ f \circ \phi$  does. If that is the case then*

$$I(\psi \circ f \circ \phi, q_0) = I(f, p_0) \cdot \text{sgn} \det[D\phi(q_0)]. \quad \square$$

## 7 Normal form of a swallowtail

In this section we show that there is a one-to-one correspondence between the sign of a swallowtail and its normal form.

**Theorem 7.1.** *Assume that  $f : (\mathbb{R}^3, p_0) \rightarrow (\mathbb{R}^3, f(p_0))$  is a simple swallowtail. Then  $f$  takes on an  $S_{1_3}$  singularity transversely at  $p_0$ . Moreover there exists an orientation preserving diffeomorphism  $\phi : (\mathbb{R}^3, \mathbf{0}) \rightarrow (\mathbb{R}^3, p_0)$  and a diffeomorphism  $\psi : (\mathbb{R}^3, f(p_0)) \rightarrow (\mathbb{R}^3, \mathbf{0})$  such that  $\psi \circ f \circ \phi = f_+$  (resp.  $f_-$ ) if and only if  $I(f, p_0) = +1$  (resp.  $-1$ ).*

*Proof.* One may assume that  $p_0 = f(p_0) = \mathbf{0}$ . Since  $\text{rank}[Df(\mathbf{0})] = 2$ , there exist coordinate systems centered at  $\mathbf{0}$  such that  $f$  has the form  $f = (h(x, y, z), y, z)$ , and  $\frac{\partial h}{\partial x}(\mathbf{0}) = 0$ .

Then  $J = \frac{\partial h}{\partial x}$ , and one may take  $K = (1, 0, 0)$ . Therefore

$$X = \frac{\partial^2 h}{\partial x^2}, \quad Y = \frac{\partial^3 h}{\partial x^3}.$$

As the mapping  $f$  is a simple swallowtail at the origin, then

$$\frac{\partial h}{\partial x}(\mathbf{0}) = \frac{\partial^2 h}{\partial x^2}(\mathbf{0}) = \frac{\partial^3 h}{\partial x^3}(\mathbf{0}) = 0,$$

and the vectors  $\nabla \left( \frac{\partial h}{\partial x} \right) (\mathbf{0})$ ,  $\nabla \left( \frac{\partial^2 h}{\partial x^2} \right) (\mathbf{0})$ ,  $\nabla \left( \frac{\partial^3 h}{\partial x^3} \right) (\mathbf{0})$  are linearly independent.

By [4, page 176], the mapping  $f$  takes on an  $S_{1_3}$  singularity transversely at  $\mathbf{0}$ . By [10] or [4, Theorem 4.1], there are diffeomorphisms  $\phi, \psi : (\mathbb{R}^3, \mathbf{0}) \rightarrow (\mathbb{R}^3, \mathbf{0})$  such that  $\psi \circ f \circ \phi = f_+$ .

By Example 3.1, there is an orientation reversing diffeomorphism  $\phi_-$  such that  $f_+ \circ \phi_- = f_-$ . Hence, if  $\phi$  does reverse the orientation then  $\phi \circ \phi_-$  does preserve it, and  $\psi \circ f \circ (\phi \circ \phi_-) = f_-$ .

Then we may assume that  $\psi \circ f \circ \phi = f_{\pm}$  and  $\phi$  preserves the orientation. By Example 3.10 and Theorem 6.2

$$\pm 1 = I(f_{\pm}, \mathbf{0}) = I(\psi \circ f \circ \phi, \mathbf{0}) = I(f, \mathbf{0}).$$

Hence  $I(f, \mathbf{0}) = +1$  if and only if  $\psi \circ f \circ \phi = f_+$ , and  $I(f, \mathbf{0}) = -1$  if and only if  $\psi \circ f \circ \phi = f_-$ .  $\square$

As an immediate consequence of Theorem 6.2 we get

**Theorem 7.2.** *Assume that  $f : (\mathbb{R}^3, p_0) \rightarrow (\mathbb{R}^3, f(p_0))$ ,  $g : (\mathbb{R}^3, p_1) \rightarrow (\mathbb{R}^3, g(p_1))$  are simple swallowtails.*

*Then  $I(f, p_0) = I(g, p_1)$  (resp.  $I(f, p_0) = -I(g, p_1)$ ) if and only if there exists an orientation preserving (resp. reversing) diffeomorphism  $\phi : (\mathbb{R}^3, p_1) \rightarrow (\mathbb{R}^3, p_0)$  and a diffeomorphism  $\psi : (\mathbb{R}^3, f(p_0)) \rightarrow (\mathbb{R}^3, g(p_1))$  such that  $\psi \circ f \circ \phi = g$ .  $\square$*

## 8 Polynomial mappings $\mathbb{R}^3 \rightarrow \mathbb{R}^3$

First we recall a method for counting the number of real zeros of an ideal in the multivariate case.

Let  $I \subset \mathbb{R}[x_1, \dots, x_n]$  be an ideal such that  $\mathcal{A} = \mathbb{R}[x_1, \dots, x_n]/I$  is an algebra of finite dimension, so that the set  $V(I)$  of real zeros of  $I$  is finite. For  $u \in \mathbb{R}[x_1, \dots, x_n]$  denote by  $t(u)$  the trace of the linear endomorphism  $\mathcal{A} \ni a \mapsto u \cdot a \in \mathcal{A}$ . Then  $t : \mathcal{A} \rightarrow \mathbb{R}$  is a linear functional.

**Theorem 8.1** ([1, 13]). *Take  $g \in \mathbb{R}[x_1, \dots, x_n]$ . Let  $\Theta$  (resp.  $\Psi$ ) be the quadratic form on  $\mathcal{A}$  given by  $\Theta(h) = t(h^2)$  (resp.  $\Psi(h) = t(g \cdot h^2)$ ), where  $h \in \mathcal{A}$ . Then*

$$\begin{aligned} \sigma(\Theta) &= \#V(I), \\ \sigma(\Psi) &= \sum \text{sgn}(g(p)), \text{ where } p \in V(I), \end{aligned}$$

and  $\sigma(\cdot)$  denotes the signature of a quadratic form.

Moreover, if  $\Psi$  is non-degenerate then  $g(p) \neq 0$  at each  $p \in V(I)$ . If that is the case then

$$(\sigma(\Theta) + \sigma(\Psi))/2 = \sum \text{sgn}(g(p)), \text{ where } p \in V(I) \cap \{g > 0\},$$

$$(\sigma(\Theta) - \sigma(\Psi))/2 = \sum \text{sgn}(g(p)), \text{ where } p \in V(I) \cap \{g < 0\}.$$

In the remainder of this section we shall show how to apply the above result so as to compute the number of positive and negative swallowtails of an polynomial mapping  $f = (f_1, f_2, f_3) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ .

Let  $J = \det[Df]$ , and let  $I_1$  be the ideal in  $\mathbb{R}[x, y, z]$  generated by  $J, \partial J/\partial x, \partial J/\partial y, \partial J/\partial z$ . Then  $J^{-1}(0)$  is the set of critical points of  $f$ .

**Proposition 8.2.** *Assume that  $I_1 = \mathbb{R}[x, y, z]$ . Then  $\nabla J(p) \neq 0$  at each  $p \in J^{-1}(0)$ , so that  $J^{-1}(0)$  is either empty or a smooth surface.*

*Proof.* Because  $1 \in I_1$ , then  $J$  and  $\nabla J$  do not vanish simultaneously at any point.  $\square$

Let  $K_{ij} = \nabla f_i \times \nabla f_j$ , where  $1 \leq i < j \leq 3$ , and let  $X_{ij} = \langle \nabla J, K_{ij} \rangle$ . Let  $I_2 \subset \mathbb{R}[x, y, z]$  denote the ideal generated by  $J$  and all  $2 \times 2$ -minors of the derivative matrix  $Df$ .

**Proposition 8.3.** *Assume that  $I_1 = I_2 = \mathbb{R}[x, y, z]$ . Then*

(i)  *$\text{rank}[Df(p)] = 2$  at any  $p \in J^{-1}(0)$ , and  $J^{-1}(0) = S_1(f)$  is either empty or a smooth surface,*

(ii) *at each  $p \in J^{-1}(0)$ , at least one  $K_{ij}(p) \neq 0$ ,*

*Proof.* By Proposition 8.2, the set  $J^{-1}(0)$  is a smooth surface. Take  $p \in J^{-1}(0)$ . Since  $I_2 = \mathbb{R}[x, y, z]$ , there is at least one non-zero  $2 \times 2$ -minor of  $Df(p)$ . Hence  $\text{rank}[Df(p)] = 2$ , i.e.  $p \in S_1(f)$ , and at least one vector field  $K_{ij}$  does not vanish at  $p$ .  $\square$

Let  $I_3$  denote the ideal generated by  $J, X_{1,2}, X_{1,3}, X_{2,3}$  and all  $2 \times 2$ -minors of the derivative matrix  $D(J, X_{1,2}, X_{1,3}, X_{2,3})$ .

**Proposition 8.4.** *Assume that  $I_1 = I_2 = I_3 = \mathbb{R}[x, y, z]$ . Then*

(i)  *$S_{1,1}(f) = J^{-1}(0) \cap \bigcap X_{ij}^{-1}(0)$  is either empty or a smooth curve,*

(ii) the set of critical points of  $f|_{S_{1,1}(f)}$  equals  $S_{1,1,1}(f)$ .

*Proof.* Take any  $p \in J^{-1}(0)$ . One may assume that  $K_{1,2}(p) \neq \mathbf{0}$ . Every  $K_{ij} \in \text{Ker } Df$  and  $\dim \text{Ker}(Df) \equiv 1$  along  $S_1(f)$ . Therefore we may apply results of Section 3 and Section 4, where  $K = K_{1,2}$ ,  $K_1 = K_{ij}$ ,  $X_1 = X_{ij}$ ,  $\alpha \equiv 1$ , so that  $J_1 = J$ . By Corollary 4.1,

$$J^{-1}(0) \cap X_{1,2}^{-1}(0) \subset J^{-1}(0) \cap \bigcap_{1 \leq i < j \leq 3} X_{ij}^{-1}(0).$$

The opposite inclusion is obvious. Therefore  $J^{-1}(0) \cap \bigcap X_{ij}^{-1}(0) = J^{-1}(0) \cap X_{1,2}^{-1}(0)$  near  $p \in J^{-1}(0)$ .

Suppose that  $p \in J^{-1}(0) \cap \bigcap X_{ij}^{-1}(0)$ , and  $K_{1,2}(p) \neq \mathbf{0}$ . As  $I_3 = \mathbb{R}[x, y, z]$ , then

$$\text{rank } D(J, X_{1,2}, X_{1,3}, X_{2,3})(p) \geq 2.$$

According to Lemma 4.4(i), each  $\nabla X_{ij}(p)$  is a linear combination of  $\nabla J(p), \nabla X_{1,2}(p)$ . Hence  $\text{rank } D(J, X_{1,2}, X_{1,3}, X_{2,3})(p) = \text{rank } D(J, X_{1,2})(p) = 2$ . By Remark 3.5,  $S_{1,1}(f) = J^{-1}(0) \cap X_{1,2}^{-1}(0) = J^{-1}(0) \cap \bigcap X_{ij}^{-1}(0)$  is a smooth curve near any  $p \in J^{-1}(0)$ . By Remark 3.6, the set of critical points of  $f|_{S_{1,1}(f)}$  equals  $S_{1,1,1}(f)$ .  $\square$

Let  $Y_{ij} = \langle \nabla X_{ij}, K_{ij} \rangle$ ,  $H_{ij} = (J, X_{ij}, Y_{ij})$ ,  $Z_{ij} = \langle \nabla Y_{ij}, K_{ij} \rangle$ , and  $g_{ij} = Z_{ij} \cdot \det[DH_{ij}]$ . Then  $J, X_{ij}, Y_{ij}, Z_{ij}, g_{ij} \in \mathbb{R}[x, y, z]$ .

Let  $I$  denote the ideal in  $\mathbb{R}[x, y, z]$  generated by  $J$ , all  $X_{ij}$ , and all  $Y_{ij}$ . Let  $\mathcal{A} = \mathbb{R}[x, y, z]/I$ . Put  $g = \sum \alpha_{ij} \cdot g_{ij}$ , where all  $\alpha_{ij}$  are non-negative, and at least one is positive. If  $\dim_{\mathbb{R}} \mathcal{A} < \infty$  then the quadratic forms  $\Theta, \Psi$  on  $\mathcal{A}$  given by

$$\Theta(h) = t(h^2) \quad , \quad \Psi(h) = t(g \cdot h^2)$$

are defined, as well as their signatures.

**Theorem 8.5.** *Assume that  $I_1 = I_2 = I_3 = \mathbb{R}[x, y, z]$  and  $\dim_{\mathbb{R}} \mathcal{A} < \infty$ . Then  $S_{1,1,1}(f) = V(I)$  is finite, and*

$$\sigma(\Theta) = \# S_{1,1,1}(f) \quad .$$

*If the quadratic form  $\Psi$  is non-degenerate, then all points in  $S_{1,1,1}(f)$  are simple swallowtails and*

$$\sigma(\Psi) = \sum I(f, p), \quad \text{where } p \in S_{1,1,1}(f).$$

If that is the case then

$$\#\{p \in S_{1,1,1}(f) \mid I(f, p) = +1\} = (\sigma(\Theta) + \sigma(\Psi)) / 2,$$

$$\#\{p \in S_{1,1,1}(f) \mid I(f, p) = -1\} = (\sigma(\Theta) - \sigma(\Psi)) / 2.$$

*Proof.* Take  $p \in S_{1,1}(f)$ . By Proposition 8.3 and 8.4, we may assume that  $K_{1,2}(p) \neq \mathbf{0}$  and  $p \in J^{-1}(0) \cap \bigcap X_{ij}^{-1}(0)$ . By Remark 3.6, the mapping  $f|_{S_{1,1}(f)}$  has a critical point at  $p$  if and only if  $p \in J^{-1}(0) \cap X_{1,2}^{-1}(0) \cap Y_{1,2}^{-1}(0)$ . Applying Corollary 4.1 and similar arguments as in the proof of Proposition 8.4, one may show that  $J^{-1}(0) \cap X_{1,2}^{-1}(0) \cap Y_{1,2}^{-1}(0) = J^{-1}(0) \cap \bigcap X_{ij}^{-1}(0) \cap \bigcap Y_{ij}^{-1}(0)$  near  $p$ .

Hence the set  $S_{1,1,1}(f)$  of critical points of  $f|_{S_{1,1}(f)}$  equals  $V(I)$ . As  $\dim \mathcal{A} < \infty$ , the set  $V(I)$  is finite, and by Theorem 8.1

$$\#S_{1,1,1}(f) = \#V(I) = \sigma(\Theta).$$

Suppose that  $\Psi$  is non-degenerate. Take any  $p \in V(I)$ . By Theorem 8.1, at least one  $\alpha_{ij} \cdot g_{ij} = \alpha_{ij} \cdot Z_{ij} \cdot \det[DH_{ij}]$  does not vanish at  $p$ . By Corollary 3.8,  $p$  is a simple swallowtail. By Proposition 4.4, each  $g_{ij}(p)$  is either zero or its sign equals  $I(f, p)$ , so that  $g(p)$  has the same sign as  $I(f, p)$ . Thus

$$\sigma(\Psi) = \sum \operatorname{sgn}(g(p)) = \sum I(f, p),$$

where  $p \in S_{1,1,1}(f)$ . The last assertion is obvious.  $\square$

**Example 8.6.** Let  $f = (-x^2y + z, y^2 + x, x^2yz + z^2 + y)$ . Applying methods presented in Theorem 8.5, with the help of SINGULAR [3], one may check that  $I_1 = I_2 = I_3 = \mathbb{R}[x, y, z]$ ,  $\dim_{\mathbb{R}} \mathcal{A} = 27$ ,  $\sigma(\Theta) = 1$ , the quadratic form  $\Psi$  is non-degenerate and  $\sigma(\Psi) = -1$ . Hence  $S_{1,1,1}(f)$  consists of one negative swallowtail.

Take  $u \in \mathbb{R}[x, y, z]$ . Let  $\Phi_1, \Phi_2$  denote quadratic forms on  $\mathcal{A}$  given by

$$\Phi_1(h) = t(u \cdot h^2) \quad , \quad \Phi_2(h) = t(u \cdot g \cdot h^2).$$

Applying the method for counting real points inside a real-algebraic constraint region presented in [13] one gets

**Theorem 8.7.** Assume that  $I_1 = I_2 = I_3 = \mathbb{R}[x, y, z]$  and  $\dim_{\mathbb{R}} \mathcal{A} < \infty$ .

If the quadratic forms  $\Psi, \Phi_1, \Phi_2$  are non-degenerate, then all points in  $S_{1,1,1}(f)$  are simple swallowtails and

$$\#\{p \in S_{1,1,1}(f) \mid u(p) > 0\} = (\sigma(\Theta) + \sigma(\Phi_1)) / 2,$$

$$\#\{p \in S_{1,1,1}(f) \mid I(f, p) = +1, u(p) > 0\} = (\sigma(\Theta) + \sigma(\Psi) + \sigma(\Phi_1) + \sigma(\Phi_2)) / 4,$$

$$\#\{p \in S_{1,1,1}(f) \mid I(f, p) = -1, u(p) > 0\} = (\sigma(\Theta) - \sigma(\Psi) + \sigma(\Phi_1) - \sigma(\Phi_2)) / 4.$$

**Example 8.8.** Let  $f = (-y - 2z - xy - xz, -2x - 2y + 3xy + z^2, z + 2y - x^2)$ , and  $u = 9 - x^2 - y^2 - z^2$ . Applying methods presented in Theorem 8.7, with the help of SINGULAR [3], one may check that  $I_1 = I_2 = I_3 = \mathbb{R}[x, y, z]$ ,  $\dim_{\mathbb{R}} \mathcal{A} = 7$ ,  $\sigma(\Theta) = 3$ , the quadratic forms  $\Psi, \Phi_1, \Phi_2$  are non-degenerate and  $\sigma(\Psi) = 1$ ,  $\sigma(\Phi_1) = 1$ ,  $\sigma(\Phi_2) = 3$ .

Hence  $S_{1,1,1}(f)$  consists of three swallowtails, two of them are positive and one is negative. In the ball  $\{u > 0\}$  there are two positive swallowtails.

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